More Properties of Det

A is invertible ⟷ $\det(A) \neq 0$

Properties of Determinants

- Basic Property 1: $det(I) = 1$
- Basic Property 2: Exchange rows reverse the sign of det
  - If a matrix A has 2 equal rows, det A = 0
- Basic Property 3: Determinant is "linear" for each row

$$det\left(\begin{bmatrix} ta & tb \\ c & d \end{bmatrix}\right) = t\,det\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right)$$

$$det\left(\begin{bmatrix} a + a' & b + b' \\ c & d \end{bmatrix}\right) = det\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) + det\left(\begin{bmatrix} a' & b' \\ c & d \end{bmatrix}\right)$$

  - A row of zeros, det A = 0
  - "Subtract k x row i from row j" does not change det

Determinants for Upper Triangular Matrix

$$U = \begin{bmatrix} d_1 & \cdots & * \\ \vdots & \ddots & \vdots \\ 0 & \cdots & d_n \end{bmatrix}$$

Killing everything above

Does not change the det

$$det(U) = det\left(\begin{bmatrix} d_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & d_n \end{bmatrix}\right)$$

Property 1

3-a $= d_1 d_2 \cdots d_n det\left(\begin{bmatrix} 1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 1 \end{bmatrix}\right)$

= 1

$det(U) = d_1 d_2 \cdots d_n$ (Products of diagonal)

Determinant v.s. Invertible

A is invertible $\Longleftrightarrow$ det(A) ≠ 0

$$A \xrightarrow[\text{Elementary row operation}]{} R$$

$det(A)$

$det(R)$

$= \pm k_1 k_2 \cdots det(A)$

Exchange: Change sign

Scaling: Multiply k

Add row: nothing

If A is invertible, R is identity

$det(R) = 1 \Rightarrow det(A) \neq 0$

If A is not invertible, R has zero row

$det(R) = 0 \Rightarrow det(A) = 0$

Invertible

We collect one more properties for invertible!

- Let A be an n x n matrix. A is invertible if and only if

onto

  - The columns of A span R^n
  - For every b in R^n, the system Ax=b is consistent
  - The rank of A is n

One-on-one

  - The columns of A are linear independent
  - The only solution to Ax=0 is the zero vector
  - The nullity of A is zero
  - The reduced row echelon form of A is I_n
  - A is a product of elementary matrices
  - There exists an n x n matrix B such that BA = I_n
  - There exists an n x n matrix C such that AC = I_n
  - det(A) $\neq$ 0

Example

A is invertible $\Longleftrightarrow$ det(A) ≠ 0

$$A = \begin{bmatrix} 1 & -1 & 2 \\ -1 & 0 & \color{red}{c} \\ 2 & 1 & 7 \end{bmatrix}$$

For what scalar c is the matrix not invertible?

det(A) = 0

$$detA = \quad 1 \cdot 0 \cdot 7 \quad +(-1) \cdot \color{red}{c} \cdot 2 \quad +2 \cdot (-1) \cdot 1$$

$$-2 \cdot 0 \cdot 2 \quad -(-1) \cdot (-1) \cdot 7 \quad -1 \cdot \color{red}{c} \cdot 1$$

$$= 0 - 2c - 2 - 7 - c \;= -3c - 9$$

not invertible $\Rightarrow$ $-3c - 9 = 0$ $\Rightarrow$ $c = -3$

More Properties of Determinants

$$det(A+B) \neq det(A)+det(B)$$

- $det(AB) = det(A)det(B)$

  Q: find $det(A^{-1})$

  $$\because A^{-1}A = I \quad \therefore det(A^{-1})det(A) = det(I) = 1$$

  $$\therefore det(A^{-1}) = 1/det(A)$$

  Q: find $det(A^2)$

  $$det(A^2) = det(A)det(A) = det(A)^2$$

- $det(A^T) = det(A)$
  - Zero row → zero column
  - Same row → same column ……

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More Properties of Determinants

- $det(AB) = det(A)det(B)$
- Proof:

If A is not invertible:

A is not invertible $\Rightarrow$ AB is not invertible

$\Rightarrow$ det AB = 0

A is not invertible $\Rightarrow$ det A = 0

$\Rightarrow$ det A det B = 0

More Properties of Determinants

- $det(AB) = det(A)det(B)$
- Proof:

If A is invertible:

$$A = E_k \cdots E_2 E_1$$

You have to proof that det EA = det E det A

(E is elementary matrix)

You have to proof that det EA = det E det A

Exchange the 2nd and 3rd rows

$$E_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

$$detE_1A = -detA$$
$$= detE_1detA$$
$$detE_1 = -1$$

Multiply the 2nd row by -4

$$E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -4 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$detE_2A = -4detA$$
$$= detE_2detA$$
$$detE_2 = -4$$

Adding 2 times row 1 to row 3

$$E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}$$

$$detE_3A = detA$$
$$= detE_3detA$$
$$detE_3 = 1$$

More Properties of Determinants

- $det(AB) = det(A)det(B)$
- Proof:

If A is invertible:

$$A = E_k \cdots E_2 E_1$$

You have to proof that det EA = det E det A

(E is elementary matrix)

$$det(A) = det(E_k) \cdots det(E_2)det(E_1)$$

$$\begin{aligned} det(A)det(B) &= det(E_k) \cdots det(E_2)det(E_1)det(B) \\ &= det(E_k) \cdots det(E_2)det(E_1 B) \\ &= det(E_k \cdots E_2 E_1 B) = det(AB) \end{aligned}$$

More Properties of Determinants

- $\det A = \det A^T$
- Proof:

A is not invertible $\Rightarrow$ $\det A = 0$

$\parallel$

A^T is not invertible $\Rightarrow$ $\det A^T = 0$

A is invertible $\det E = \det E^T$ …… in the textbook

det E = det E^T …… in the textbook

Exchange the 2nd and 3rd rows

$$E_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} = \mathrm{E_1}^{\mathrm{T}} \qquad det\mathrm{E_1} = det\mathrm{E_1}^{\mathrm{T}}$$

Multiply the 2nd row by -4

$$E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -4 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \mathrm{E_2}^{\mathrm{T}} \qquad det\mathrm{E_2} = det\mathrm{E_2}^{\mathrm{T}}$$

Adding 2 times row 1 to row 3

$$E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix} \quad \mathrm{E_3}^{\mathrm{T}} = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad det\mathrm{E_3} = det\mathrm{E_3}^{\mathrm{T}}$$

More Properties of Determinants

det E = det E^T …… in the textbook

- det A = det A^T
- Proof:

A is invertible $\quad A = E_k \cdots E_2 E_1$

$$det(A) = det(E_k) \cdots det(E_2) det(E_1)$$

$$A^T = (E_k \cdots E_2 E_1)^T = {E_1}^T {E_2}^T \cdots {E_k}^T$$

$$det(A^T) = det\left({E_1}^T\right) det\left({E_2}^T\right) \cdots det\left({E_k}^T\right)$$

$$= det(E_1) det(E_2) \cdots det(E_k)$$

More Properties of Determinants

det E = det E^T …… in the textbook

- det A = det A^T
- Proof:

$$det(A) = sum\ of\ n!\ terms$$

Format of each term: $a_{1\alpha}a_{2\beta}a_{3\gamma}\cdots a_{n\omega}$

Sorted by column indices

Find an element in each row

permutation of 1,2, …, n

Format of each term: $a_{\alpha'1}a_{\beta'2}a_{\gamma'3}\cdots a_{\omega'n}$

Find an element in each column

permutation of 1,2, …, n

A v.s. A^T

- Rank A = Rank A^T
- det A = det A^T