More Properties of Det

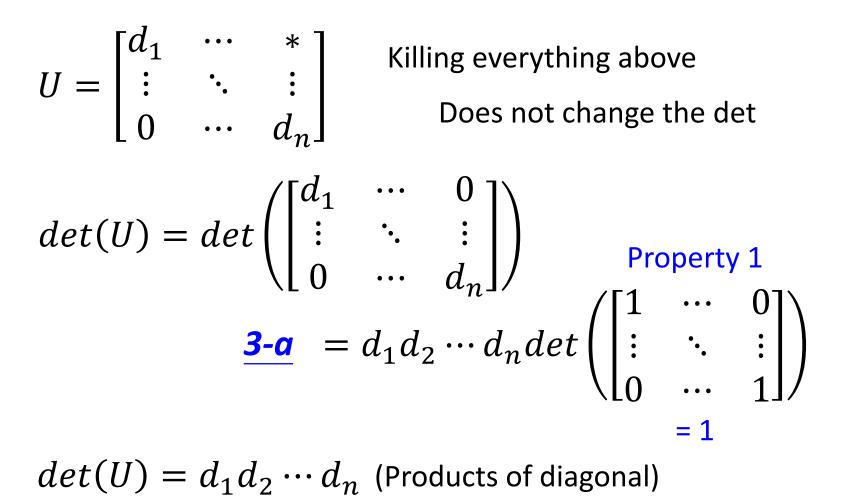


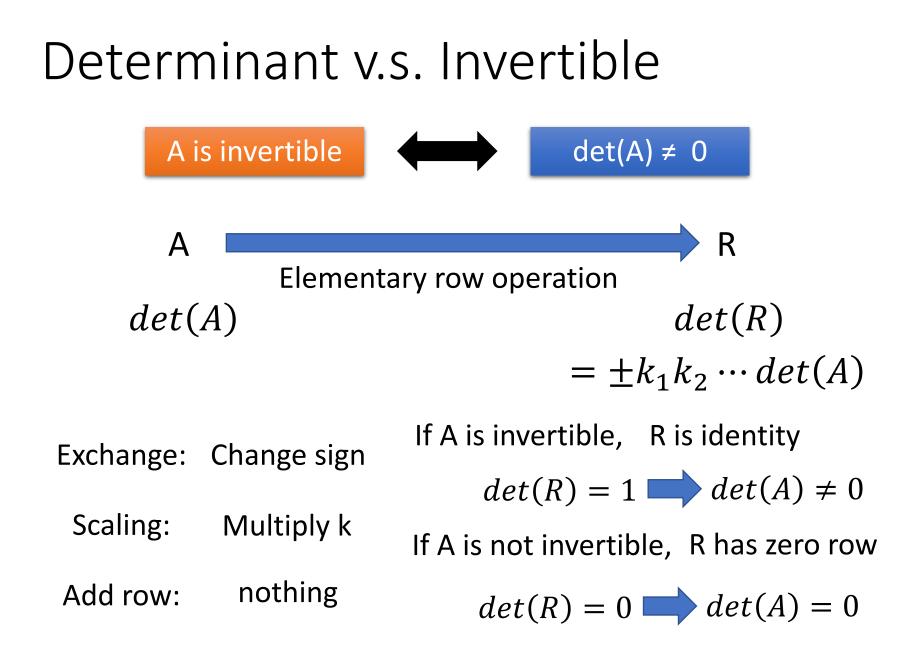
- Basic Property 1: det(I) = 1
- Basic Property 2: Exchange rows reverse the sign of det
 - If a matrix A has 2 equal rows, det A = 0
- Basic Property 3: Determinant is "linear" for each row

$$det \left(\begin{bmatrix} ta & tb \\ c & d \end{bmatrix} \right) = tdet \left(\begin{bmatrix} a & b \\ c & d \end{bmatrix} \right)$$
$$det \left(\begin{bmatrix} a + a' & b + b' \\ c & d \end{bmatrix} \right) = det \left(\begin{bmatrix} a & b \\ c & d \end{bmatrix} \right) + det \left(\begin{bmatrix} a' & b' \\ c & d \end{bmatrix} \right)$$

- A row of zeros, det A = 0
- "Subtract k x row i from row j" does not change det

Determinants for Upper Triangular Matrix

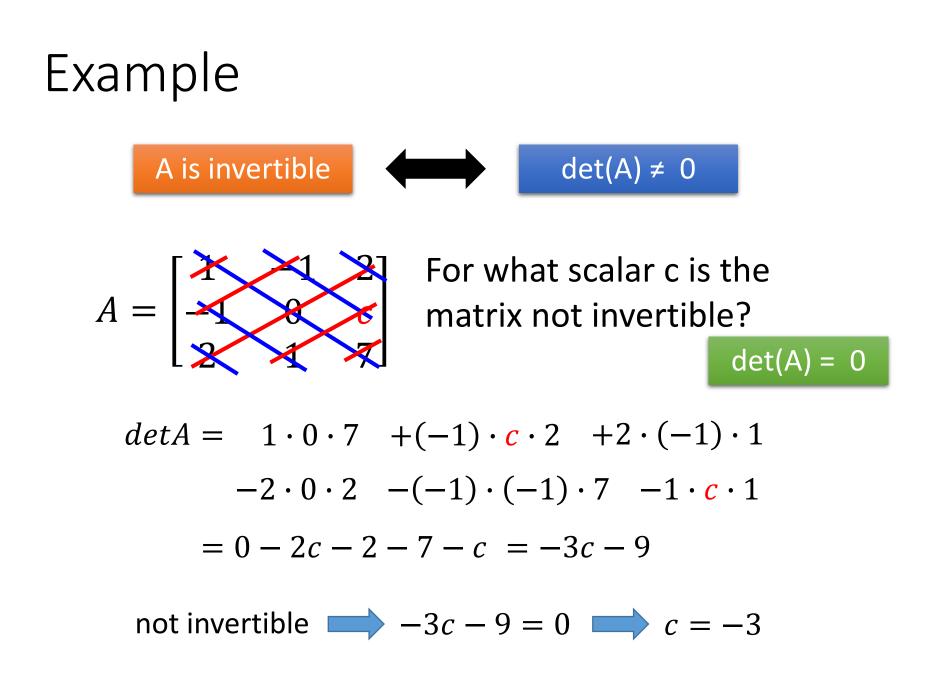




Invertible

We collect one more properties for invertible!

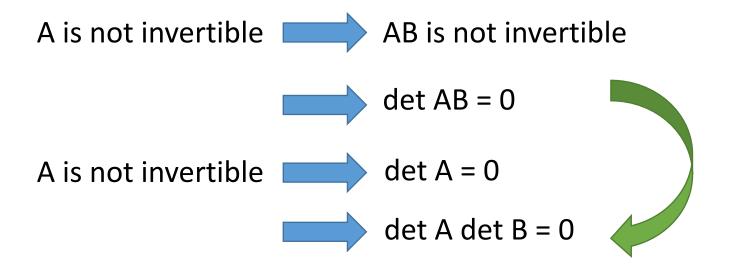
- Let A be an n x n matrix. A is invertible if and only if
 - The columns of A span Rⁿ
- For every b in Rⁿ, the system Ax=b is consistent
 - The rank of A is n
 - The columns of A are linear independent
- One-
- on-one
- The only solution to Ax=0 is the zero vector
- The nullity of A is zero
- The reduced row echelon form of A is I_n
- A is a product of elementary matrices
- There exists an n x n matrix B such that $BA = I_n$
- There exists an n x n matrix C such that AC = In
- det(A) ≠ 0



- det(AB) = det(A)det(B)Q: find $det(A^{-1})$ $\therefore A^{-1}A = I \quad \therefore det(A^{-1})det(A) = det(I) = 1$ $\therefore det(A^{-1}) = 1/det(A)$ Q: find $det(A^2)$ $det(A^2) = det(A)det(A) = det(A)^2$
- $det(A^T) = det(A)$
 - Zero row \rightarrow zero column
 - Same row → same column

P212 - 215

- det(AB) = det(A)det(B)
- Proof:
 - If A is not invertible:



- det(AB) = det(A)det(B)
- Proof:

If A is invertible:

 $A = E_k \cdots E_2 E_1$

You have to proof that det EA = det E det A

(E is elementary matrix)

You have to proof that det EA = det E det A

Exchange the 2 nd and 3 rd rows	$detE_1A = -detA$
$E_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$	$= detE_1 detA$ $detE_1 = -1$
Multiply the 2 nd row by -4	$detE_2A = -4detA$
$E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -4 & 0 \\ 0 & 0 & 1 \end{bmatrix}$	$= detE_2 detA$ $detE_2 = -4$
Adding 2 times row 1 to row 3	$detE_3A = detA$
$E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}$	$= detE_3 detA$ $detE_3 = 1$

- det(AB) = det(A)det(B)
- Proof:

If A is invertible:

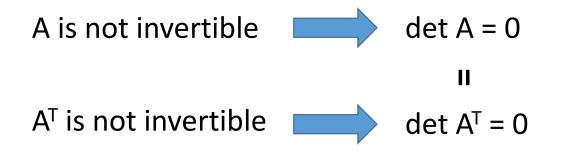
 $A = E_k \cdots E_2 E_1$

You have to proof that det EA = det E det A

(E is elementary matrix)

 $det(A) = det(E_k) \cdots det(E_2)det(E_1)$ $det(A)det(B) = det(E_k) \cdots det(E_2)det(E_1)det(B)$ $= det(E_k) \cdots det(E_2)det(E_1B)$ $= det(E_k \cdots E_2E_1B) = det(AB)$

- det A = det A^T
- Proof:



A is invertible $\det E = \det E^T$ in the textbook

Exchange the 2nd and 3rd rows

$$E_{1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} = E_{1}^{T} \qquad det E_{1} = det E_{1}^{T}$$

Multiply the 2nd row by -4

$$E_{2} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -4 & 0 \\ 0 & 0 & 1 \end{bmatrix} = E_{2}^{T} \qquad detE_{2} = detE_{2}^{T}$$

Adding 2 times row 1 to row 3

$$E_{3} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix} \quad E_{3}^{T} = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad detE_{3} = detE_{3}^{T}$$

det $E = det E^T$ in the textbook

- det A = det A^T
- Proof:

A is invertible $A = E_k \cdots E_2 E_1$ $det(A) = det(E_k) \cdots det(E_2) det(E_1)$ $A^T = (E_k \cdots E_2 E_1)^T = E_1^T E_2^T \cdots E_k^T$ $det(A^T) = det(E_1^T) det(E_2^T) \cdots det(E_k^T)$ $= det(E_1) det(E_2) \cdots det(E_k)$

det E = det E^T in the textbook

- det A = det A^T
- Proof: $det(A) = sum \ of \ n! \ terms$

Format of each term: $a_{1\alpha}a_{2\beta}a_{3\gamma}\cdots a_{n\omega}$

Sorted by column indices

Find an element in each row

permutation of 1,2, ..., n

Format of each term: $a_{\alpha'1}a_{\beta'2}a_{\gamma'3}\cdots a_{\omega'n}$

Find an element in each column

permutation of 1,2, ..., n

A v.s. A^{T}

- Rank $A = Rank A^T$
- det A = det A^T

